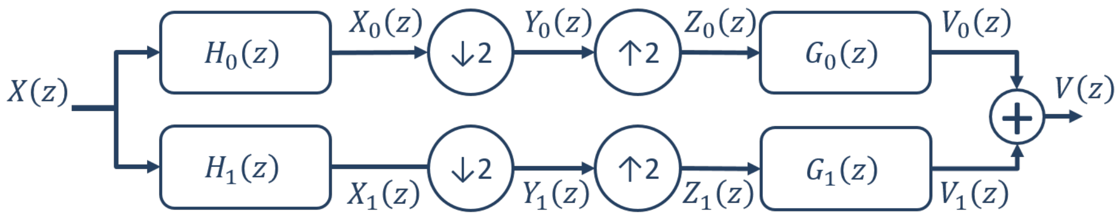
Question #1: Consider the following 2-channel filter bank shown below.



Let the filters be defined by

$$h_0[n] = \alpha\left(\delta[n] + \delta[n-1]\right) \qquad g_0[n] = \alpha\left(\delta[n] + \delta[n+1]\right)$$
$$h_1[n] = \alpha\left(\delta[n] - \delta[n-1]\right) \qquad g_1[n] = \alpha\left(\delta[n] - \delta[n+1]\right)$$

(a) Compute the Z-transform of the filter responses and find the value of α that satisfies the alias canceling filter bank conditions. Show why.

Solution:

$$H_0(z) = \alpha\left(1 + z^{-1}\right) \qquad G_0(z) = \alpha\left(1 + z^{+1}\right)$$
$$H_1(z) = \alpha\left(1 - z^{-1}\right) \qquad G_1(z) = \alpha\left(1 - z^{+1}\right)$$

The alias canceling filter bank condition is:

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2$$
$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

For the first condition:

$$\begin{aligned}
\alpha\left(1 + z^{-1}\right)\alpha\left(1 + z^{+1}\right) + \alpha\left(1 - z^{-1}\right)\alpha\left(1 - z^{+1}\right) &= 2 \\
\alpha^2\left(1 + z^{-1} + z^{+1} + 1\right) + \alpha^2\left(1 - z^{-1} - z^{+1} + 1\right) &= 2 \\
\alpha^2(1+1) + \alpha^2(1+1) &= 2 \\
\alpha^2 4 &= 2 \\
\alpha &= \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

For the second condition:

$$\begin{aligned}
\alpha\left(1 - z^{-1}\right)\alpha\left(1 + z^{+1}\right) + \alpha\left(1 + z^{-1}\right)\alpha\left(1 - z^{+1}\right) &= 0 \\
\alpha^2\left(1 - z^{-1} + z^{+1} - 1\right) + \alpha^2\left(1 + z^{-1} - z^{+1} - 1\right) &= 0 \\
0 &= 0
\end{aligned}$$

(b) With this value of α, determine all of the intermediate signals ($X_m(z)$, $Y_m(z)$, $Z_m(z)$, $V_m(z)$, and $V(z)$ for all m) in the **time domain** for excitation $X(z) = 1 + z^{-1} - z^{-2} + z^{-3}$.

Solution:

$$\begin{aligned}
X(z) &= 1 + z^{-1} - z^{-2} + z^{-3} \\
x[n] &= 1 + \delta[n-1] - \delta[n-2] + \delta[n-3]
\end{aligned}$$

$$\begin{aligned}
X_0(z) &= X(z)H_0(z) = \alpha(1 + 2z^{-1} + z^{-4}) \\
Y_0(z) &= \frac{1}{2}\left[X_0(z^{1/2}) + X_0(-z^{1/2})\right] \\
&= \frac{1}{2}\left[\alpha(1 + 2z^{-1/2} + z^{-2}) + \alpha(1 + 2(-z^{1/2})^{-1} + (-z^{1/2})^{-2})\right] \\
&= \frac{1}{2}\left[\alpha(1 + z^{-2}) + \alpha(1 + z^{-2})\right] \\
&= \alpha(1 + z^{-2}) \\
Z_0(z) &= Y_0(z^2) \\
&= \alpha(1 + z^{-4}) \\
V_0(z) &= G_0(z)Z_0(z) \\
&= \alpha\alpha\left(1 + z^{+1}\right)(1 + z^{-4}) \\
&= \alpha^2\left(1 + z^{+1} + z^{-4} + z^{-3}\right)
\end{aligned}$$

$$\begin{aligned}
X_1(z) &= X(z)H_1(z) = \alpha(1 - 2z^{-2} + 2z^{-3} - z^{-4}) \\
Y_1(z) &= \frac{1}{2}\left[X_1(z^{1/2}) + X_1(-z^{1/2})\right] \\
&= \frac{1}{2}\left[\alpha(1 - 2z^{-1/2} + z^{-2}) + \alpha(1 - 2(-z^{1/2})^{-1} + (-z^{1/2})^{-2})\right] \\
&= \alpha(1 - 2z^{-1} - z^{-2}) \\
Z_1(z) &= Y_1(z^2) \\
&= \alpha(1 - 2z^{-2} - z^{-4}) \\
V_1(z) &= G_1(z)Z_1(z) \\
&= \alpha\alpha\left(1 - z^{+1}\right)(1 - 2z^{-2} - z^{-4}) \\
&= \alpha^2\left(-z + 1 + 2z^{-1} - 2z^{-2} + z^{-3} - z^{-4}\right)
\end{aligned}$$

$$\begin{aligned}
V(z) &= V_1(z) + V_2(z) \\
&= 2\alpha^2\left(1 + z^{-1} - z^{-2} + z^{-3}\right) \\
&= 1 + z^{-1} - z^{-2} + z^{-3}
\end{aligned}$$

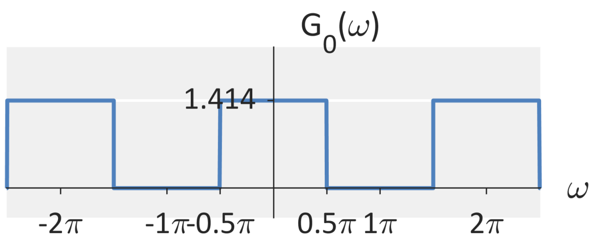
So the time-domain responses are

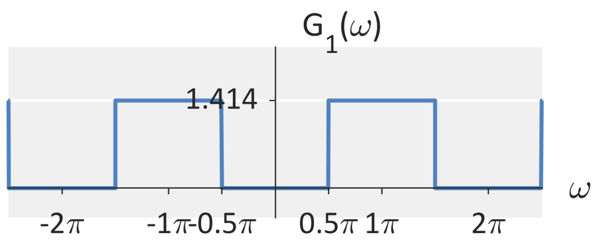
$$
\begin{aligned}
x_0[n] &= (1/\sqrt{2})(\delta[n] + 2\delta[n-1] + \delta[n-4]) \\
y_0[n] &= (1/\sqrt{2})(\delta[n] + \delta[n-2]) \\
z_0[n] &= (1/\sqrt{2})(\delta[n] + \delta[n-4]) \\
v_0[n] &= (1/2)(\delta[n+1] + \delta[n] + \delta[n-3] + \delta[n-4])
\end{aligned}
$$

and

$$
\begin{aligned}
x_1[n] &= (1/\sqrt{2})(\delta[n] - 2\delta[n-2] + 2\delta[n-3] - \delta[n-4]) \\
y_1[n] &= (1/\sqrt{2})(\delta[n] - 2\delta[n-1] + \delta[n-2]) \\
z_1[n] &= (1/\sqrt{2})(\delta[n] - 2\delta[n-2] + \delta[n-4]) \\
v_1[n] &= (1/2)(-\delta[n+1] + \delta[n] + 2\delta[n-1] - 2\delta[n-2] + \delta[n-3] - \delta[n-4]) \\
v[n] &= 1 + \delta[n-1] - \delta[n-2] + \delta[n-3]
\end{aligned}
$$

Question #2: Consider the following 2-channel filter bank shown in Question #1. Let the filters be defined by the following frequency responses





(a) Assume $H_0(z) = G_0(z^{-1})$ and $H_1(z) = G_1(z^{-1})$. Do the filters satisfy the orthogonal filter bank conditions? Show why.

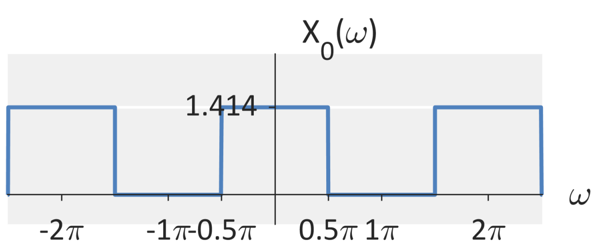
Solution: The orthogonal filter bank condition is

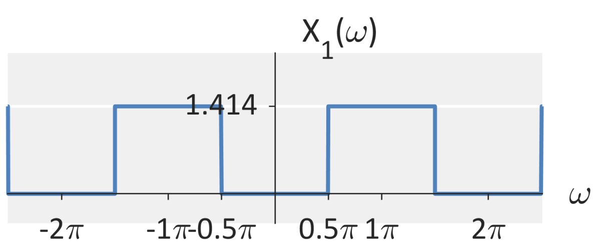
$$G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$$
$$G_1(z)G_1(z^{-1}) + G_1(-z)G_1(-z^{-1}) = 2$$
$$G_0(z)G_1(z^{-1}) + G_0(-z)G_1(-z^{-1}) = 0$$

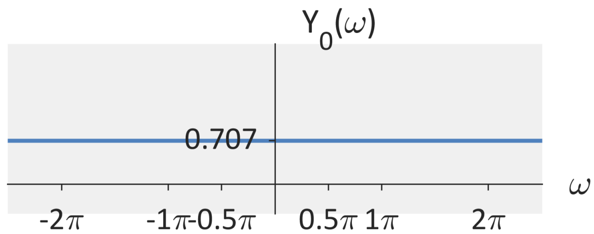
These terms become either 2 or 0 in their respective regions, which leads us to satisfying the orthogonal filter bank conditions.

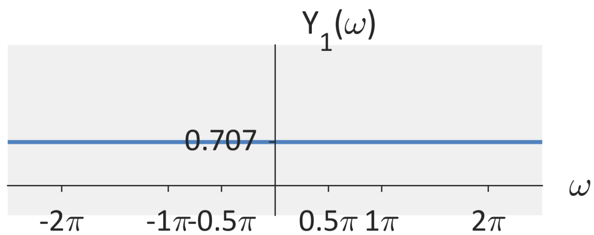
(b) Sketch all of the intermediate signals ($X_m(z)$, $Y_m(z)$, $Z_m(z)$, $V_m(z)$, and $V(z)$ for all m) in the **frequency domain** for excitation $X(z) = 1$.

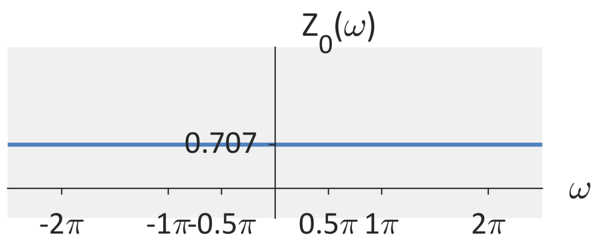
Solution:

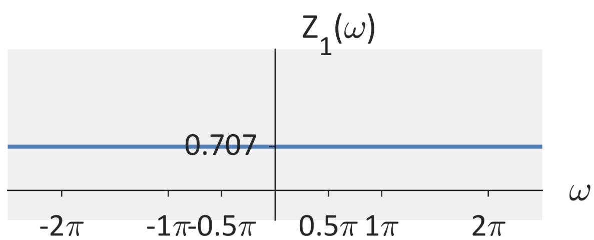


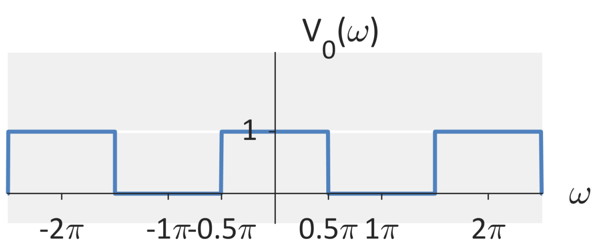


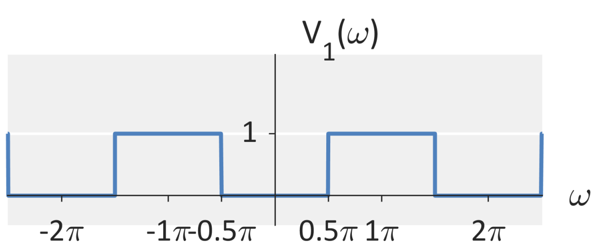

Y0(ω)
0.707
ω
-2π -1π -0.5π 0.5π 1π 2π

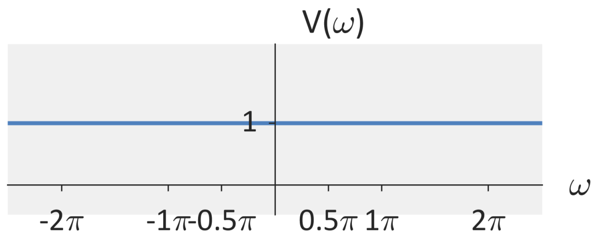

Y1(ω)
0.707
ω
-2π -1π -0.5π 0.5π 1π 2π


Z0(ω)
0.707
ω
-2π -1π -0.5π 0.5π 1π 2π


Z1(ω)
0.707
ω
-2π -1π -0.5π 0.5π 1π 2π


V0(ω)
1
ω
-2π -1π -0.5π 0.5π 1π 2π


V1(ω)
1
ω
-2π -1π -0.5π 0.5π 1π 2π



V(ω)
1
ω
-2π
-1π
-0.5π
0.5π
1π
2π