Full Name: ______________________________

EEE 5502 (Fall 2022) – Practice **Filter Properties, 2022**

Question #1: Consider an impulse response $h[n]$ for an FIR system.

(a) Let the coefficients of $h[n]$ be **even** symmetric around some value of n (i.e., it does not have to be symmetric around $n = 0$). Show that this system has a linear phase.

Solution: If we have N coefficients, we can represent them as b_k, where $0 \leq k \leq N$. Because the coefficients are even symmetric about some center, $b_k == b_{N-k}$.

Since we don't know where the coefficients are centered, we include a constant value C that shifts the impulse response to its proper location (note that C is not index of the center but the index of leftmost edge of $h[n]$).

We can thus write the impulse response like this:

$$h[n] = \sum_{k=0}^{N} b_k \delta[n - (k + C)]$$

To see what its phase looks like, we take the DTFT:

$$H(\omega) = \sum_{k=0}^{N} b_k \delta[n - (k + C)] e^{-j\omega(k+C)}$$

Given that the coefficients are symmetric ($b_k = b_{N-k}$), we can change the range of k to $0 \leq k \leq \frac{N}{2}$, with some modifications to include the exponential from both k and $N - k$:

$$H(\omega) = \sum_{k=0}^{N/2} b_k \delta[n - (k + C)](e^{-j\omega(k+C)} + e^{-j\omega(N-k+C)})$$

We can expand the exponentials and divide out a common factor:

$$H(\omega) = \sum_{k=0}^{N/2} b_k \delta[n - (k + C)](e^{-j\omega k} e^{-j\omega C} + e^{j\omega k} e^{-j\omega C} e^{-j\omega N})$$

$$H(\omega) = \sum_{k=0}^{N/2} b_k \delta[n - (k + C)](e^{-j\omega C})(e^{-j\omega k} + e^{j\omega k} e^{-j\omega N})$$

This form is slightly unpleasant, and its phase is not very obvious. To try and get only one exponential term, we can pull a factor out from the exponentials that allows us to use Euler's formula:

$$H(\omega) = \sum_{k=0}^{N/2} b_k \delta[n - (k + C)](e^{-j\omega C})(e^{-j\omega N/2})(e^{-j\omega k} e^{j\omega N/2} + e^{j\omega k} e^{-j\omega N/2})$$

Applying Euler's formula ($2\cos(x) = e^{jx} + e^{-jx}$):

$$H(\omega) = \sum_{k=0}^{N/2} b_k \delta[n - (k + C)](e^{-j\omega C})(e^{-j\omega N/2})(2\cos(k - N/2))$$

This new expression only has one complex portion, which gives us the phase.

$$(e^{-j\omega C})(e^{-j\omega N/2}) = e^{-j\omega(C+N/2)}$$

This phase is linear with a constant slope of $C + N/2$.

(b) Let the coefficients of $h[n]$ be **odd** symmetric around some value of n (i.e., it does not have to be symmetric around $n = 0$). Show that this system has a linear phase. (Note: you can just explain how your previous proof changes)

Solution: The only difference here is that now, $b_k = -b_{N-k}$. We can still perform the same steps as above, but with some sign changes. When it comes time to perform Euler's formula, we must use the form $-2j\sin(x) = e^{jx} - e^{-jx}$. This produces a fully imaginary value, which does not change the slope of the phase. The phase remains linear.