Full Name: ______________________________

EEE 5502 (Fall 2022) – Practice **Fourier Transform Properties, 2022**

Question #1: Prove the following for a signal $x(t)$ and its Fourier Transform $X(\Omega)$.

(a) If $x(t)$ is real, then $X(\Omega) = X^*(-\Omega)$, where $(\cdot)^*$ is the complex conjugate

Solution: A real signal satisfies:

$$x(t) = x^*(t)$$

So, we can do

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt \\ &= \int_{-\infty}^{\infty} x^*(t)e^{-j\Omega t}\,dt \\ &= \left[\int_{-\infty}^{\infty} x(t)e^{+j\Omega t}\,dt\right]^* \\ &= \left[\int_{-\infty}^{\infty} x(t)e^{-j(-\Omega)t}\,dt\right]^* \\ &= X^*(-\Omega) \end{aligned}$$

Hence, we get that

$$X(\Omega) = X^*(-\Omega)$$

(b) If $x(t)$ is real and even, then $X(\omega)$ is real and even
(hint: $X(\Omega)$ is even if $X(\Omega) = X(-\Omega)$ and $X(\omega)$ is real if $X(\Omega) = X^*(\Omega)$)

Solution: An even signal satisfies:

$$x(t) = x(-t)$$

So, we can do

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt \\ &= \int_{-\infty}^{\infty} x(-t)e^{-j\Omega t}\,dt \\ &\text{Perform change of variables: } \tau = -t \\ &= \left[\int_{-\infty}^{\infty} x(\tau)e^{j\Omega\tau}\,dt\right]^* \\ &= \left[\int_{-\infty}^{\infty} x(\tau)e^{-j(-\Omega)\tau}\,dt\right]^* \\ &= X(-\Omega) \end{aligned}$$

Hence, we get that $X(\omega)$ is even.

If we also apply $X(\Omega) = X^*(-\Omega)$, we get

$$\begin{aligned} X(\Omega) &= X(-\Omega) \\ &= X^*(\Omega) \end{aligned}$$

Hence, we get that $X(\omega)$ is real.

(c) If $x(t)$ is real and odd, then $X(\Omega)$ is imaginary and odd

Solution: An odd signal satisfies:

$$x(t) = -x(-t)$$

Also, a signal is imaginary if

$$\begin{gathered} X^*(\Omega) = -X(\Omega) \\ \text{or} \\ X(\Omega) = -X^*(\Omega) \end{gathered}$$

So, we can do

$$\begin{aligned}
X(\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt \\
&= \int_{-\infty}^{\infty} -x(-t)e^{-j\Omega t}\,dt \\
&\text{Perform change of variables: } \tau = -t \\
&= \left[\int_{-\infty}^{\infty} -x(\tau)e^{j\Omega\tau}\,dt\right]^* \\
&= \left[\int_{-\infty}^{\infty} -x(\tau)e^{-j(-\Omega)\tau}\,dt\right]^* \\
&= -X(-\Omega)
\end{aligned}$$

Hence, we get that $X(\omega)$ is odd.

If we also apply $X(\Omega) = X^*(-\Omega)$, we get

$$\begin{aligned}
X(\Omega) &= -X(-\Omega) \\
&= -X^*(\Omega)
\end{aligned}$$

Hence, we get that $X(\omega)$ is imaginary.

Question #2: Use the definitions of the discrete-time Fourier transform (DTFT) / inverse DTFT to answer the questions below. As a reminder, the DTFT / inverse DTFT is defined by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad , \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\, d\omega$$

(a) Show that the DTFT of a signal is always periodic with a period of 2π.

Solution:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(\omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n}$$

$$X(\omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi n}$$

$$X(\omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}(1) \quad since \quad n \in \mathbb{Z}$$

$$X(\omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(\omega)$$

Therefore $X(\omega)$ is 2π periodic

(b) Show that if $x[n]$ is real, then its DTFT $X(\omega)$ has conjugate symmetry:

$$X(\omega) = X^*(-\omega)$$

Solution:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(-\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}$$

$$X^*(-\omega) = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n}$$

$$Real \quad signal: \quad X[n] = X^*[n]$$

$$X^*(-\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X^*(-\omega) = X(\omega)$$

(c) Show that if $x[n]$ is real (the complex case is not much more difficult), then

$$X_e(\omega) = \sum_{n=-\infty}^{\infty} x_e[n]\cos(\omega n) \quad , \quad X_o(\omega) = -j\sum_{n=-\infty}^{\infty} x_o[n]\sin(\omega n)$$

where $x_e[n]$ and $x_o[n]$ are the even and odd parts of $x[n]$, respectively.

Solution:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n])e^{j\omega n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n])(\cos\omega n - j\sin\omega n)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x_e[n](\cos\omega n - j\sin\omega n) + \sum_{n=-\infty}^{\infty} x_o[n](\cos\omega n - j\sin\omega n)$$

recall that sum of odd functions go to zero

$$X(\omega) = \sum_{n=-\infty}^{\infty} x_e[n]\cos\omega n - j\sum_{n=-\infty}^{\infty} x_o[n]\sin\omega n$$

$$X(\omega) = X_e(\omega) + X_o(\omega)$$

Question #3: Rectangular functions are used throughout signal processing and other fields (e.g., an aperture of a camera in two-dimensions can be represented by a rectangle and its effect on images is a convolution). Hence, the CTFT or DTFT of the rectangular function is also commonly used. In this problem, let's derive the DTFT of a rectangular function. That is, show that following DTFT pair is true:

$$x[n] = u[n+N] - u[n-N-1]$$
$$X(\omega) = \frac{\sin(\omega(N+1/2))}{\sin(\omega/2)}$$

You may use the DTFT tables to show this, except do not use the row of the table that explicitly gives this relationship. [1]

Solution:

$$x[n] = u[n+N] - u[n-N-1]$$
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$X(\omega) = \sum_{n=-\infty}^{\infty} (u[n+N] - u[n-N-1])e^{-j\omega n}$$
$$X(\omega) = \sum_{n=-N}^{N} (u[n+N] - u[n-N-1])e^{-j\omega n}$$
$$X(\omega) = \sum_{n=-N}^{N} (1)e^{-j\omega n}$$

Sum of a geometric series with a common ratio of $e^{-j\omega}$ and length $2N+1$

$$X(\omega) = e^{j\omega N}\frac{1-e^{-j\omega(2N+1)}}{1-e^{-j\omega}}$$
$$X(\omega) = e^{j\omega N}e^{-j\omega(2N+1)/2}\frac{e^{j\omega(2N+1)/2} - e^{-j\omega(2N+1)/2}}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}$$
$$X(\omega) = e^{j\omega N}e^{-j\omega(2N+1)/2}\frac{(2j\sin(\omega(2N+1)/2)}{e^{-j\omega/2}(2j\sin(\omega/2))}$$

ignore the phase terms

$$X(\omega) = \frac{2j\sin(\omega(N+1/2)}{2j\sin(\omega/2)}$$
$$X(\omega) = \frac{\sin(\omega(N+1/2))}{\sin(\omega/2)}$$

[1] Hint: You may want to use something similar to $1 - e^{-j\omega} = e^{-j\omega/2}(e^{+j\omega/2} - e^{-j\omega/2})$