Full Name: ____________________

EEE 5502 (Fall 2022) – Practice **Continuous and Discrete-Time Systems, 2022**

Question #1: Consider the following discrete-time systems. Assume M is a positive integer. Assume the input to every system is $x[n]$ (i.e., the $y[n] = \mathcal{H}\{x[n]\}$).

$$y_1[n] = x[n-M] \qquad y_2[n] = \frac{1}{M}\sum_{m=0}^{M-1} x[n-m]$$

$$y_3[n] = \frac{1}{M}\sum_{m=1}^{M} x[n+m] \qquad y_4[n] = \sum_{m=-\infty}^{n} |x[m]|^2$$

$$y_5[n] = 0.5y_5[n-M] + x[n] \qquad y_6[n] = x[Mn]$$

(a) Identify a non-linear system. Show that it is non-linear.

Solution:

System 1 is linear.
System 2 is linear.
System 3 is linear.
System 4 is non-linear.
System 5 is linear.
System 6 is linear.

System 1 Proof:

$$\begin{aligned} &(1) \quad a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\} &&= ax_1[n-M] + bx_2[n-M] \\ &(2) \quad \mathcal{H}\{ax_1[n] + bx_2[n]\} &&= ax_1[n-M] + bx_2[n-M] \end{aligned}$$

(1) and (2) are equal, so the system is linear.

System 2 Proof:

$$\begin{aligned} &(1) \quad a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\} &&= a\frac{1}{M}\sum_{m=0}^{M-1} x_1[n-m] + b\frac{1}{M}\sum_{m=0}^{M-1} x_2[n-m] \\ &(2) \quad \mathcal{H}\{ax_1[n] + bx_2[n]\} &&= \frac{1}{M}\sum_{m=0}^{M-1} ax_1[n-m] + bx_2[n-m] \end{aligned}$$

(1) and (2) are equal, so the system is linear.

System 3 Proof:

$$\begin{aligned} &(1) \quad a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\} &&= a\frac{1}{M}\sum_{m=1}^{M} x_1[n+m] + b\frac{1}{M}\sum_{m=0}^{M-1} x_2[n+m] \\ &(2) \quad \mathcal{H}\{ax_1[n] + bx_2[n]\} &&= \frac{1}{M}\sum_{m=1}^{M} ax_1[n+m] + bx_2[n+m] \end{aligned}$$

(1) and (2) are equal, so the system is linear.

System 4 Proof:

$$\begin{aligned}
&(1) \qquad a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\} &&= a\sum_{m=-\infty}^{n} |x_1[m]|^2 + b\sum_{m=-\infty}^{n} |x_2[m]|^2 \\
&(2) \qquad \mathcal{H}\{ax_1[n] + bx_2[n]\} &&= \sum_{m=-\infty}^{n} |ax_1[m] + bx_2[m]|^2
\end{aligned}$$

(1) and (2) are not equal, so the system is not linear.

System 5 Proof:

$$\begin{aligned}
&(1) \qquad a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\} &&= a(0.5y_1[n-M] + x_1[n]) + b(0.5y_2[n-M] + x_2[n]) \\
& &&= a(0.25y_1[n-2M] + 0.5x_1[n-M] + x_1[n]) \\
& &&\quad + b(0.25y_2[n-2M] + 0.5x_2[n-M] + x_2[n]) \\
&(2) \qquad \mathcal{H}\{ax_1[n] + bx_2[n]\} &&= 0.5\,\mathcal{H}\{ax_1[n-M] + bx_2[n-M]\} + (ax_1[n] + bx_2[n]) \\
& &&= 0.25\,\mathcal{H}\{ax_1[n-2M] + bx_2[n-2M]\} \\
& &&\quad + 0.5(ax_1[n-M] + bx_2[n-M])) + (ax_1[n] + bx_2[n])
\end{aligned}$$

(1) and (2) are equal (by induction), so the system is linear.

System 6 Proof:

$$\begin{aligned}
&(1) \qquad a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\} &&= ax_1[Mn] + bx_2[Mn] \\
&(2) \qquad \mathcal{H}\{ax_1[n] + bx_2[n]\} &&= ax_1[Mn] + bx_2[Mn]
\end{aligned}$$

(1) and (2) are equal, so the system is linear.

(b) Identify a time-varying system. Show that it is time-varying.

Solution: System 1 is time-invariant.
System 2 is time-invariant.
System 3 is time-invariant.
System 4 is time-invariant.
System 5 is time-invariant.
System 6 is time-varying.

System 1 Proof:

$$\begin{aligned}
&(1) \qquad \mathcal{H}\{x[n-N]\} &&= x_1[n-N-M] \\
&(2) \qquad y_1[n-N] &&= x_1[(n-N)-M]
\end{aligned}$$

(1) and (2) are equal, so the system is time-invariant.

System 2 Proof:

$$
\begin{aligned}
&(1) \qquad \mathcal{H}\left\{x[n-N]\right\} && = \frac{1}{M}\sum_{m=0}^{M-1} x[n-m-N] \\
&(2) \qquad y_2[n-N] && = \frac{1}{M}\sum_{m=0}^{M-1} x[(n-N)-m]
\end{aligned}
$$

(1) and (2) are equal, so the system is time-invariant.

System 3 Proof:

$$
\begin{aligned}
&(1) \qquad \mathcal{H}\left\{x[n-N]\right\} && = \frac{1}{M}\sum_{m=1}^{M} x[n+m-N] \\
&(2) \qquad y_3[n-N] && = \frac{1}{M}\sum_{m=1}^{M} x[(n-N)+m]
\end{aligned}
$$

(1) and (2) are equal, so the system is time-invariant.

System 4 Proof:

$$
\begin{aligned}
&(1) \qquad \mathcal{H}\left\{x[n-N]\right\} && = \sum_{m=-\infty}^{n} |x[m-N]|^2 \\
&(2) \qquad y_4[n-N] && = \sum_{m=-\infty}^{n-N} |x[m]|^2 \\
& && \text{change of variables: } m = r-N \\
& && = \sum_{r-N=-\infty}^{n-N} |x[r-N]|^2 \\
& && = \sum_{r=-\infty}^{n} |x[r-N]|^2
\end{aligned}
$$

(1) and (2) are equal, so the system is time-invariant.

System 5 Proof: Note that

$$
\begin{aligned}
y_5[n] &= 0.5(0.5y[n-2M] + x[n-M]) + x[n] \\
&= 0.25y[n-2M] + 0.5x[n-M]) + x[n] \\
&= x[n] + 0.5x[n-M] + 0.25x[n-2M] + \cdots
\end{aligned}
$$

$$\begin{aligned}
(1) \quad & \mathcal{H}\{x[n-N]\} && = x[n-N] + 0.5x[n-M-N] + 0.25x[n-2M-N] + \cdots \\
(2) \quad & y_5[n-N] && = x[n-N] + 0.5x[(n-N)-N] + 0.25x[(n-N)-2M] + \cdots
\end{aligned}$$

(1) and (2) are equal, so the system is time-invariant.

System 6 Proof:

$$\begin{aligned}
(1) \quad & \mathcal{H}\{x[n-N]\} && = x_1[Mn-N] \\
(2) \quad & y_1[n-N] && = x_1[M(n-N)]
\end{aligned}$$

(1) and (2) are not equal, so the system is not time-invariant.

(c) Identify a non-causal system. Show that it is non-causal.

Solution: System 1 is causal.
System 2 is causal.
System 3 is non-causal (output $y_3[n]$ depends on future inputs, such as $x[n+M]$).
System 4 is causal.
System 5 is causal.
System 6 is non-causal.

(d) Identify a memoryless system. Show that it is memoryless.

Solution: Trick question. No systems are memoryless. System 1 is not memoryless.
System 2 is not memoryless.
System 3 is not memoryless.
System 4 is not memoryless.
System 5 is not memoryless.
System 6 is not memoryless.

(e) Identify a non-BIBO stable system. Show that it is not BIBO stable

Solution: System 1 is BIBO stable.
System 2 is BIBO stable.
System 3 is BIBO stable.

System 4 is not BIBO stable (the output of $u[n]$ is go to infinity).
System 5 is BIBO stable.
System 6 is BIBO stable.

Question #2: Consider the following discrete-time systems. Assume $M = 4$ is a positive integer. Assume the input to every system is $x[n]$ (i.e., the $y[n] = \mathcal{H}\{x[n]\}$).

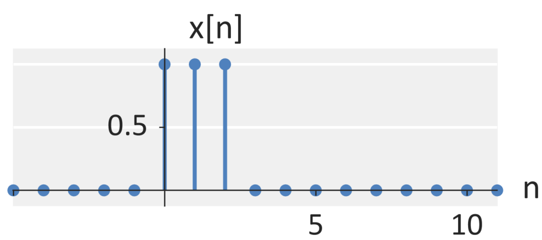
$$y_1[n] = x[n - M] \qquad y_2[n] = \frac{1}{M}\sum_{m=0}^{M-1} x[n-m]$$

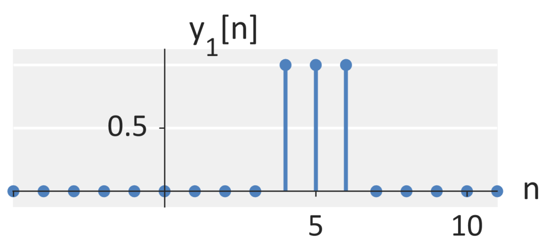
$$y_3[n] = \frac{1}{M}\sum_{m=1}^{M} x[n+m] \qquad y_4[n] = \sum_{m=-\infty}^{n} |x[n]|^2$$

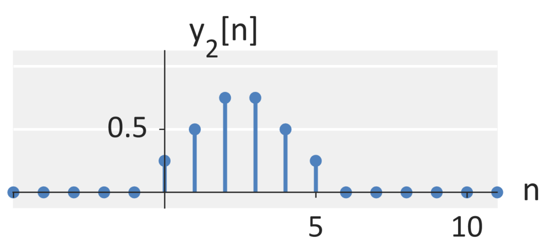
$$y_5[n] = 0.5y[n - M] + x[n] \qquad y_6[n] = x[Mn]$$

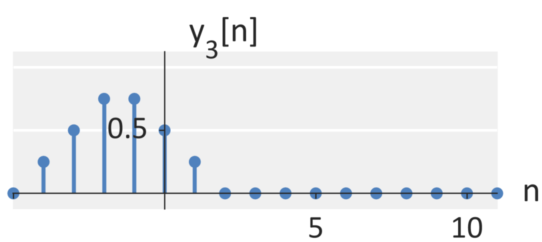
Sketch the output of each system for input $x[n] = u[n] - u[n-3]$ for $-5 \leq n \leq 11$.

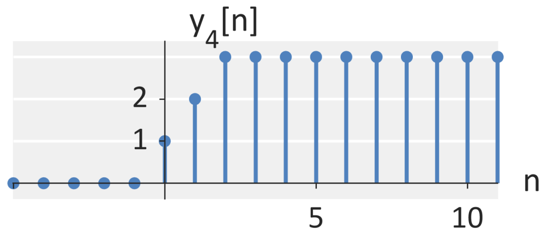
Solution:



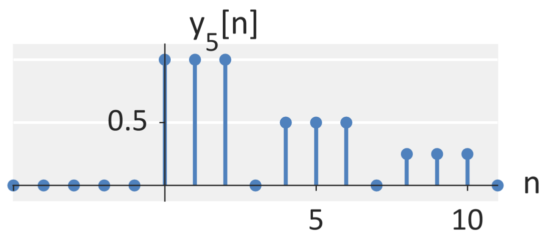




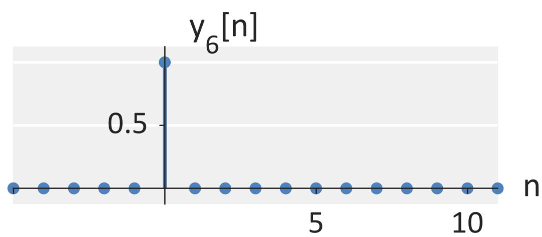




y4[n]
2
1
n
5
10



y5[n]
0.5
n
5
10



y6[n]
0.5
n
5
10