Multirate Digital Signal Processing

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Overview

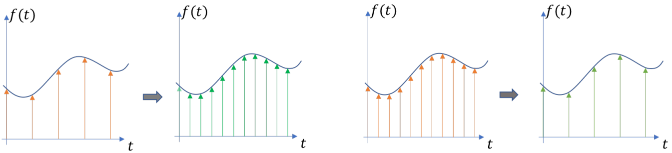
1. An Introduction to Multirate Digital Signal Processing
   - What is Multirate Digital Signal Processing?
   - Why Multirate DSP?
2. Decimation and Interpolation by an Integral Factor
   - Decimation
   - Interpolation

What is Multirate DSP?

The following two definitions (from Proakis & Manolakis) best define Multirate Digital Signal Processing.

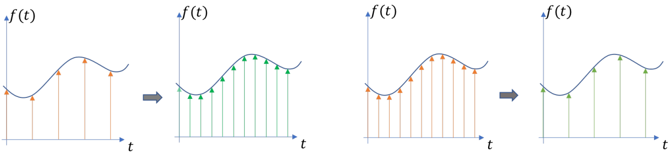
Sampling Rate Conversion: The process of converting a digital signal from a given sampling rate to a different sampling rate is called *Sampling Rate Conversion*.

Multirate DSP Systems: Systems that employ multiple sampling rates in the processing of digital signals are called *Multirate Digital Signal Processing Systems*.



Why Multirate DSP?

1. Sampling rate conversion in Communication Systems where the receivers and transmitter may have a different sampling rate.
2. Signals can be acquired from different sources sampled at different sample rates – for processing the signals to make decisions the best way is to bring them all to a common sampling rate



Multirate DSP

Suppose that, we have the values $f[0], f[1], f[2], f[3], \cdots$ sampled with sampling period T_x from a signal $f(t)$. This situation is depicted below:

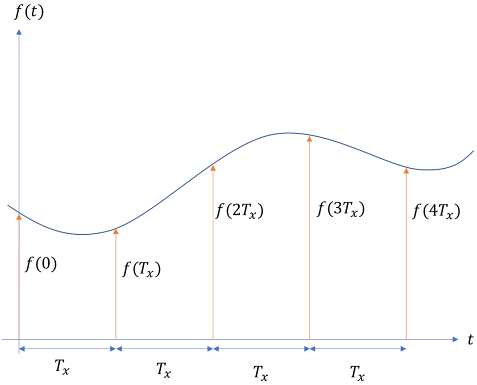


Figure: Continuous Time signal $f(t)$ sampled at $f_x = \frac{1}{T_x}$

Multirate DSP

Say, on sampling the same signal $f(t)$ with a sampling period $T_y (\neq T_x)$ we have something like below:

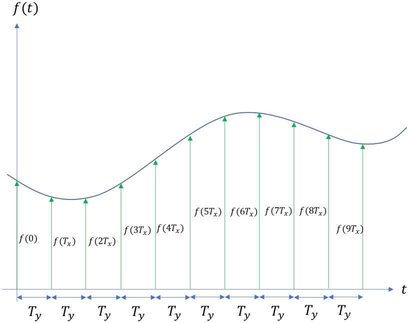
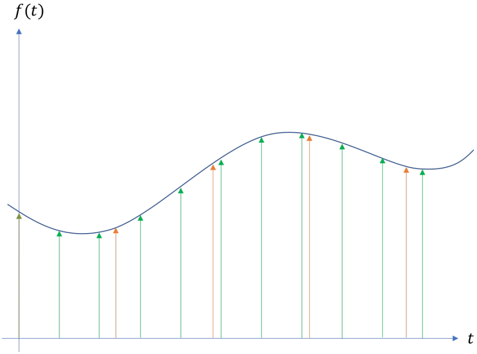


Figure: Continuous Time Signal $f(t)$ sampled at $f_y = \frac{1}{T_y}$

In the picture above, we illustrated $T_x > T_y$, but the other $T_y > T_x$ can also be true.

Let's state what we want to do:



Given the values of the function at the orange locations in the above picture, we want to predict the values at the green locations.

An Intutitive Way

1. An intuitive way of thinking about this is by reconstructing the continuous time signal and sampling it at the required rate (f_y).
2. To reconstruct the signal, we first pass it through a low pass filter with cut-off frequency f_x.
3. From Sampling theory, the highest frequency in the reconstructed signal is at most $\frac{f_x}{2}$.

Before we sample again at sampling rate f_y, we need to consider two cases:

1. $f_y > f_x$
2. $f_x > f_y$

Multirate DSP

$f_y > f_x$

In this case, since,

$$f_y > f_x \Rightarrow f_y > 2\left(\frac{f_x}{2}\right) \quad (1)$$

The Nyquist Sampling Condition is satisfied, therefore, we can sample at the rate f_y with no aliasing effects.

$f_x > f_y$

In this case, we see that,

$$f_y < 2\left(\frac{f_x}{2}\right) \quad (2)$$

The Nyquist Sampling condition is not satisfied, therefore to prevent aliasing, we first use an anti-aliasing filter (with cut off frequency $\frac{f_y}{2}$) before reconstuction, so that the maximum frequency of the signal is $\frac{f_y}{2}$.

We perform Multirate operations on a given discrete time signal $x[n]$, sampled from a continuous time signal at a sampling frequency of f_x to get a new sequence $y[n]$ which is a sampled version of the same continuous time signal sampled at a different rate, f_y. In this class we study two special cases:

1. **Decimation by a Factor D** (a special case of $f_x < f_y$ where $f_y = \frac{f_x}{D}$)
   Given a discrete time signal sampled at f_x, we want to find the discrete time signal sampled from the same continuous time signal sampled at $\frac{f_x}{D}$. We do this in two steps.
2. **Interpolation by a Factor L** (a special case of $f_x > f_y$ where $f_y = Lf_x$)
   Given a discrete time signal sampled at f_x, we want to find the discrete time signal sampled from the continuous time signal sampled at Lf_x.

Multirate DSP - Decimation

Using an Impulse train - First step

We decimate (kill!) $D-1$ samples between 0 and D, i.e. we set them all to zero. We continue doing this to all samples between kD to $(k+1)D$, for all $k = 1, 2, \cdots$. Mathematically, this can be done by multiplying the signal $x[n]$ with an impulse train of the form:

$$p[n] = \begin{cases} 1, & \text{if n is a multiple of D} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Since $p[n]$ is periodic with period D, we use Discrete Fourier Series to write $p[n]$ as:

$$p[n] = \frac{1}{D} \sum_{k=0}^{D-1} e^{j\frac{2\pi}{D}kn} \tag{4}$$

We soon see why this is convenient.

Discrete Time Fourier Series allows us to write any periodic function as a linear combination of complex sinusoids:

$$p[n] = \frac{1}{D}\sum_{k=0}^{D-1} c_k e^{j\frac{2\pi}{D}kn} \tag{5}$$

where c_k is given by,

$$c_k = \sum_{n=0}^{D-1} p[n] e^{-j\frac{2\pi}{D}kn} \tag{6}$$

We find c_k for $k = 0, 1, 2, \cdots, D-1$ corresponding to $p[n]$.

Multirate DSP - Decimation

By substituting $p[n]$ in the above, for each $k = 0, 1, 2, \cdots, D-1$, we have,

$$c_k = 1.e^{-j\frac{2\pi}{D}k(0)} = 1 \tag{7}$$

Substituting the above back in 4 we have,

$$p[n] = \frac{1}{D}\sum_{k=0}^{D-1} e^{j\frac{2\pi}{D}kn} \tag{8}$$

The signal we obtain after first step is:

$$x[n]p[n] = \frac{1}{D}\sum_{k=0}^{D-1} x[n]e^{j\frac{2\pi}{D}kn} \tag{9}$$

Multirate DSP - First step - Example Decimation by a factor of 3

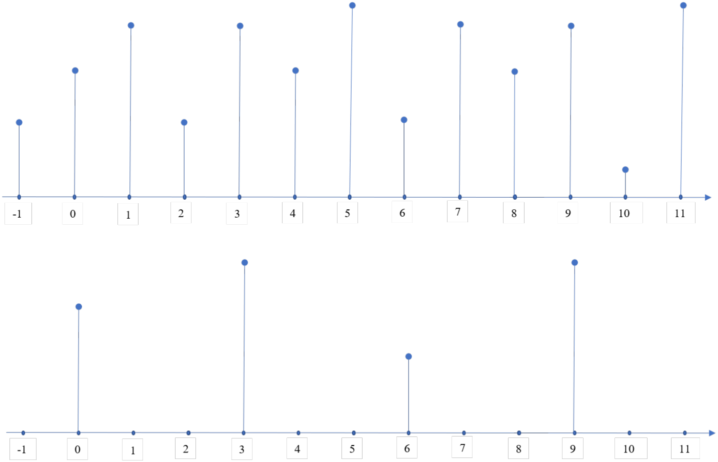


Figure: Obtaining $x[n]p[n]$ from $x[n]$

Multirate DSP - Decimation

After the first step, we downsample the signal.

Downsampling by a factor D

For a discrete time signal $x[n]$, the following mathematical expression best describes downsampling by a factor D,

$$\tilde{x}[m] = x[mD] \tag{10}$$

In this, $\tilde{x}[m]$ is the downsampled signal and the process of going from $x[n]$ to $\tilde{x}[m]$ is called Decimation by a factor of D.

Multirate DSP - Second Step - Example Decimation by a factor of 3

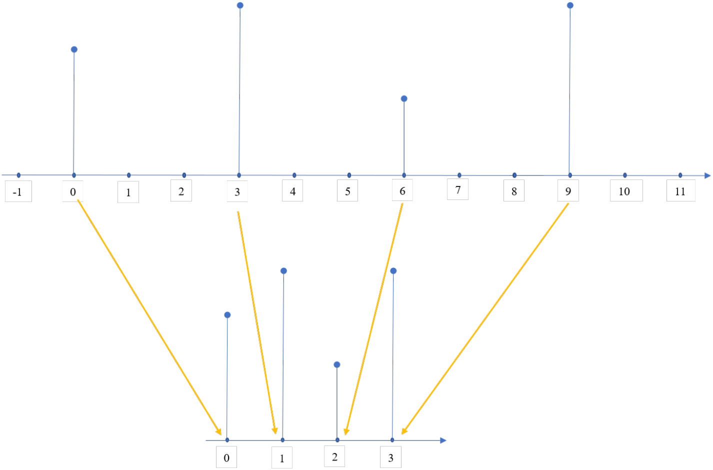


Figure: Obtaining $\tilde{x}[n]$ from $x[n]p[n]$

In this section, we want to relate the z-domain expressions of the signal before and after Decimation by a factor D. Let $X(z)$ be the z-transform of $x[n]$. We, therefore, can write,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{11}$$

And let, $Y(z)$ be the z-transform of $\tilde{x}[m]$. We can write,

$$Y(z) = \sum_{m=-\infty}^{\infty} \tilde{x}[m]z^{-m} = \sum_{m=-\infty}^{\infty} x[mD]p[mD]z^{-m} \tag{12}$$

Now we use equation (9) to help modify the above:

$$Y(z) = \sum_{r=-\infty}^{\infty} x[r] \frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi k \frac{r}{D}} z^{-r/D} \tag{13}$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} \sum_{r=-\infty}^{\infty} x[r] e^{j2\pi k \frac{r}{D}} z^{-r/D} = \frac{1}{D} \sum_{k=0}^{D-1} \sum_{r=-\infty}^{\infty} x[r] \left(z e^{-j2\pi k} \right)^{-r/D} \tag{14}$$

Thus, we have,

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(z^{\frac{1}{D}} e^{-j\frac{2\pi k}{D}} \right) \tag{15}$$

We put, $z = e^{j\omega}$ to analyze in the frequency domain.

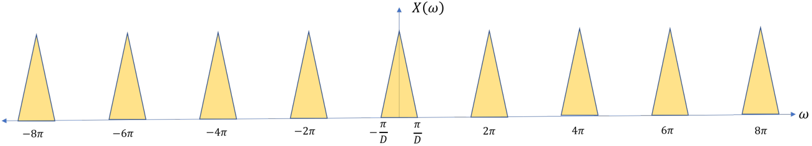
$$z^{\frac{1}{D}} e^{-j\frac{2\pi k}{D}} = e^{j\frac{\omega}{D}} e^{-j\frac{2\pi k}{D}} = e^{j\frac{\omega - 2\pi k}{D}} \tag{16}$$

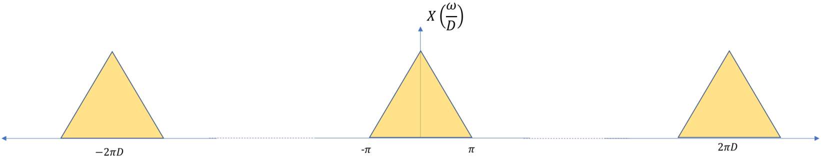
Thus,

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - 2\pi k}{D}\right) \tag{17}$$

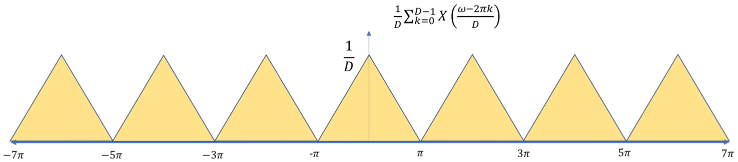
Multirate DSP - Frequency Domain Analysis

Let us start with a bandlimited signal bandlimited to digital angular frequency $\frac{\pi}{D}$.





Multirate DSP - Frequency Domain Analysis



From the above, when $-\pi \leq \omega \leq \pi$, we can see that,

$$\frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega-2\pi k}{D}\right) = \frac{1}{D}X\left(\frac{\omega}{D}\right) \tag{18}$$

Moreover, it is also periodic with period 2π, therefore it is a valid Discrete Time Fourier Transform.

Multirate DSP - Frequency Domain Analysis - Decimation

When we have an input signal of frequency more than $\frac{\pi}{D}$, we first pass it through a low pass filter with cut off frequency $\frac{\pi}{D}$ and then do the two steps shown above. We do this to prevent aliasing, therefore, we call the low pass filter an anti-aliasing filter. Thus the final structure of the decimation process will look like:

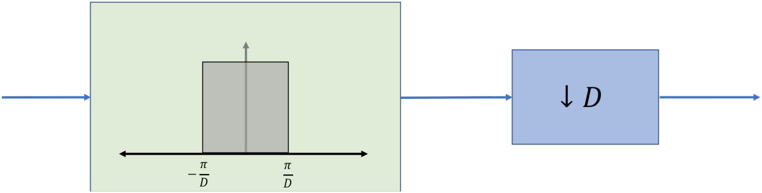


Figure: Decimation by a factor D

Interpolating with Zeros - First Step

We modify the given signal $x[n]$ by placing L-1 zeros between every two samples. Mathematically, we create a new function $\bar{x}[m]$, as,

$$\bar{x}[Lm] = x[m] \tag{19}$$

for $k = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$. We set the value of 0 for all arguments which are not a multiple of L.

Multirate DSP - First Step - Example Interpolation by 4

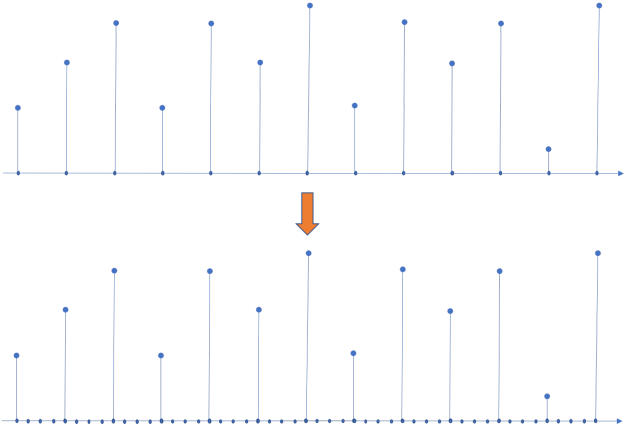


Figure: First step of Interpolation L=4

Z transform Analysis of First Step

Let, $X(z)$ be the z-transform of $x[n]$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{20}$$

Also let, $\bar{X}(z)$ be z-transform of $\bar{x}[m]$.

$$\bar{X}(z) = \sum_{m=\infty}^{\infty} \bar{x}[m]z^{-m} = \sum_{k=-\infty}^{\infty} \bar{x}[kL]z^{-kL} = \sum_{k=-\infty}^{\infty} x[k]z^{-kL} = X(z^L) \tag{21}$$

Frequency Analysis of First Step

We evaluate the frequency response by evaluating on the unit circle, $z = e^{j\omega}$. We see that $z^L = e^{j\omega L}$. Thus,

$$\bar{X}(\omega) = X(\omega L) \tag{22}$$

Multirate DSP - Interpolation

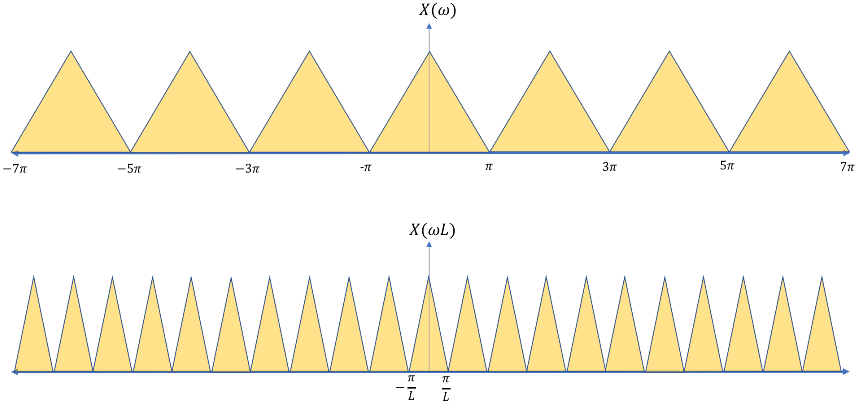


Figure: Frequency Analysis of Interpolation by L

We observe that we get L copies of the frequency response of the input signal in the interval $[-\pi, \pi]$. Therefore, we filter out everything outside $\left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$ on the interval $[-\pi, \pi]$. Thus, we use the following low-pass filter:

$$H_L(\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{L} \\ 0 & \pi \geq |\omega| \geq \frac{\pi}{L} \end{cases} \tag{23}$$

Multirate DSP - Interpolation

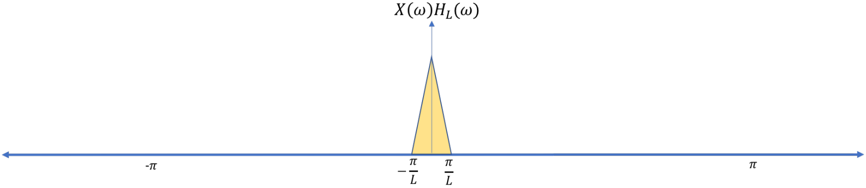


Figure: Output Discrete Time Fourier Transform

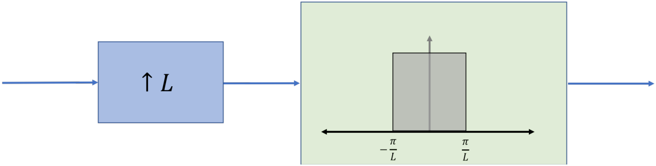


Figure: Interpolation by L

The End