Full Name: ________________________________ **ExamID: 010001**

EEL 4750 / EEE 5502 (Fall 2018) – Exam #03 **Date: Dec. 4, 2018**

Question	# of Points Possible	# of Points Obtained	Grader
# 1	17		
# 2	17		
# 3	16		
# 4	16		
# 5	18		
# 6	16		
Total	100		

For full credit when sketching: remember to label axes and make locations and amplitudes clear.

Before starting the exam, read and sign the following agreement.

By signing this agreement, I agree to solve the problems of this exam while adhering to the policies and guidelines of the University of Florida and EEL 4750 / EEE 5502 and without additional external help. The guidelines include, but are not limited to,

- The University of Florida honor pledge: "On my honor, I have neither given nor received unauthorized aid in doing this assignment."
- Only one 8.5 by 11 inch cheat sheet (double-sided) may be used
- No calculators or computers may be used
- No textbooks or additional notes may be used
- No collaboration is allowed
- No cheating is allowed

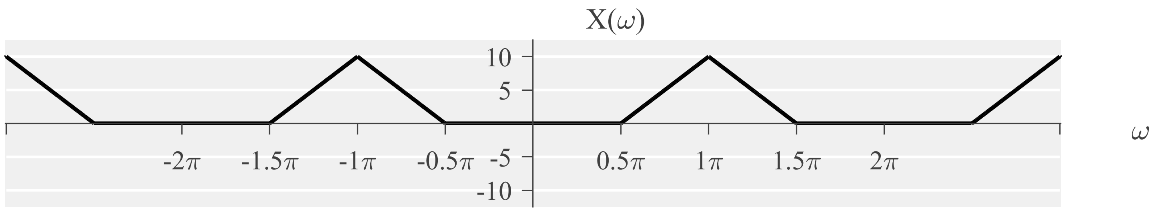
________________________________ ________________________________

Student Date

Full Name: ______________________________ **ExamID: 010001**

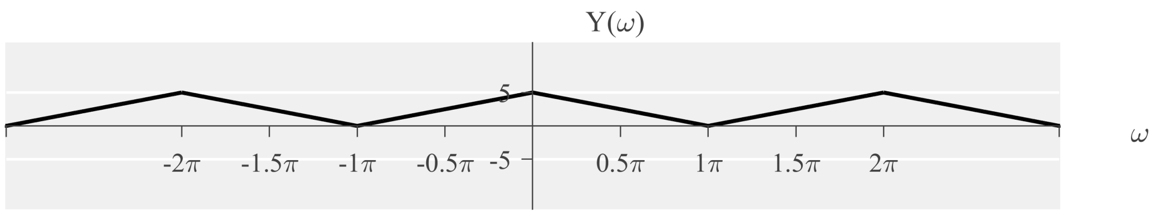
EEL 4750 / EEE 5502 (Fall 2018) – Exam #03 **Date: Dec. 4, 2018**

Question #1: Consider the DTFT of the signal $x[n]$ (i.e., $X(\omega)$) shown below.



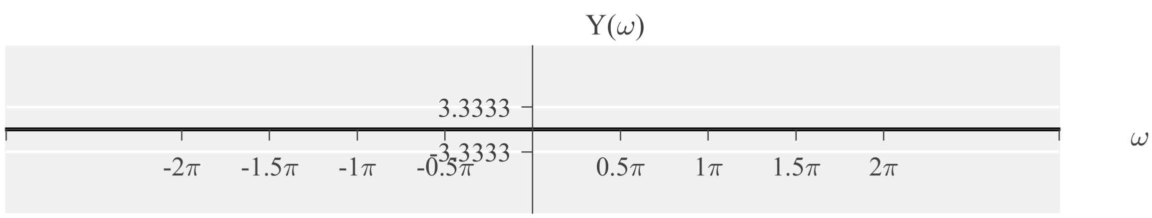
(a) *(9 pts)* Sketch the DTFT (from $\omega = -3\pi$ to $\omega = 3\pi$) of $x[n]$ after downsampling by 2 (**with no anti-aliasing filter**). Remember to label important locations / values.

Solution:

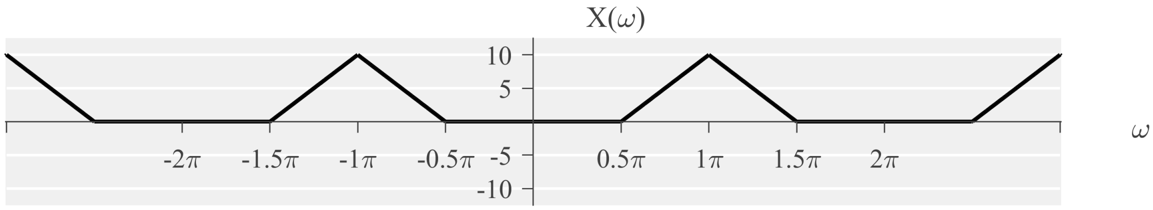


(b) *(8 pts)* Sketch the DTFT (from $\omega = -3\pi$ to $\omega = 3\pi$) of $x[n]$ after downsampling by 3 (**with an anti-aliasing filter**). Remember to label important locations / values.

Solution:

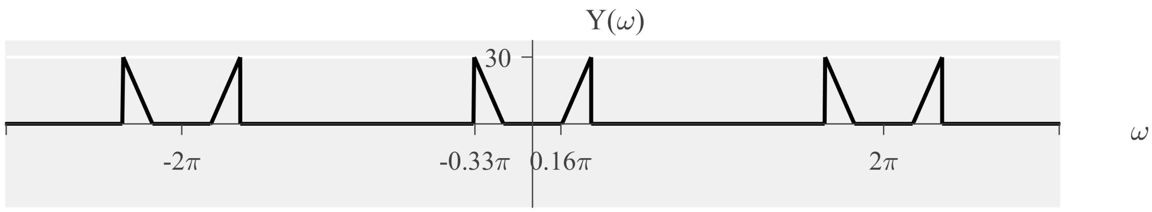


Question #2: Consider the DTFT of the signal $x[n]$ (i.e., $X(\omega)$) shown below.



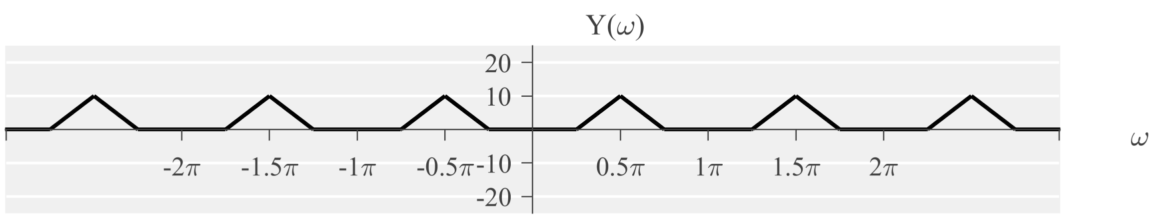
(a) *(9 pts)* Sketch the DTFT (from $\omega = -3\pi$ to $\omega = 3\pi$) of $x[n]$ after upsampling by 3 (**with an interpolation filter**). Remember to label important locations / values.

Solution:



(b) *(8 pts)* Sketch the DTFT (from $\omega = -3\pi$ to $\omega = 3\pi$) of $x[n]$ after upsampling by 2 (**with no interpolation filter**). Remember to label important locations / values.

Solution:



Question #3: Consider the desired frequency response

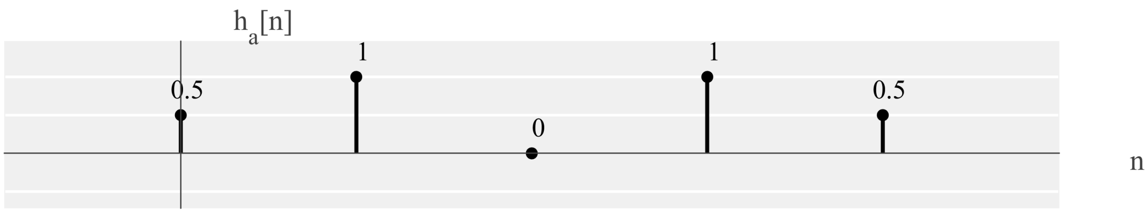
$$H_{da}(\omega) = \frac{e^{-j\omega}}{1-(1/2)e^{-j\omega}} + \frac{e^{+j\omega}}{1-(1/2)e^{+j\omega}} \qquad , \qquad H_{db}(\omega) = \frac{1}{e^{+j5\omega}+e^{-j5\omega}}$$

(a) *(8 pts)* Approximate $H_{da}(\omega)$ with a length $N = 5$ windowing method. Use a rectangular window. **Force the resulting filter to be causal and linear phase.** Sketch the time-domain filter coefficients $h_a[n]$ with these requirements.

Solution:

$$h_d[n] = (1/2)^{n-1}u[n-1] + (1/2)^{-n-1}u[-n-1]$$

After shifting by $(N-1)/2 = 2$ samples to force casually and a linear phase, the solution is:



(b) *(8 pts)* Approximate $H_{db}(\omega)$ with a length $N = 5$ frequency sampling method. **Force the resulting filter to be causal and linear phase.** Compute the time-domain filter coefficients $h_b[n]$ with these requirements.

Solution: The frequency coefficients up to π are:

$$\omega_0 = 0 \qquad H_d(0) = \frac{1}{1+1} = \frac{1}{2}$$

$$\omega_1 = \frac{2\pi(1)}{5} \qquad H_d(2\pi/5) = \frac{1}{1+1} = \frac{1}{2}$$

$$\omega_2 = \frac{2\pi(2)}{5} \qquad H_d(4\pi/5) = \frac{1}{1+1} = \frac{1}{2}$$

Therefore,

$$h_b[n] = \frac{1}{2} + \cos\left(\frac{2\pi}{5}(n-2)\right) + \cos\left(\frac{4\pi}{5}(n-2)\right) \quad \text{for} \quad 0 \le n \le 4$$

Question #4: Consider the desired filter transfer function

$$H_d(s) = \frac{5}{s-j-1} + \frac{5}{s+j-1}$$

(a) *(8 pts)* Approximate $H_d(s)$ as a discrete-time IIR filter by approximating the differential equation with a sampling rate $T = 1$. **Determine the resulting z-domain poles and zeros.**

Solution: $s \to \frac{1}{T}(1 - z^{-1})$

$$\begin{aligned}
H_a(z) &= \frac{5}{(1-z^{-1}) - j - 1} + \frac{5}{(1-z^{-1}) + j - 1} \\
&= \frac{5}{-j - z^{-1}} + \frac{5}{j - z^{-1}} \\
&= \frac{5(-j - z^{-1} + j - z^{-1})}{(-j - z^{-1})(j - z^{-1})} \\
&= \frac{-10z^{-1}}{(1+z^{-2})} = \frac{-10z}{(z^2+1)}
\end{aligned}$$

poles: $z = -j, j$
zeros: $z = 0, \infty$

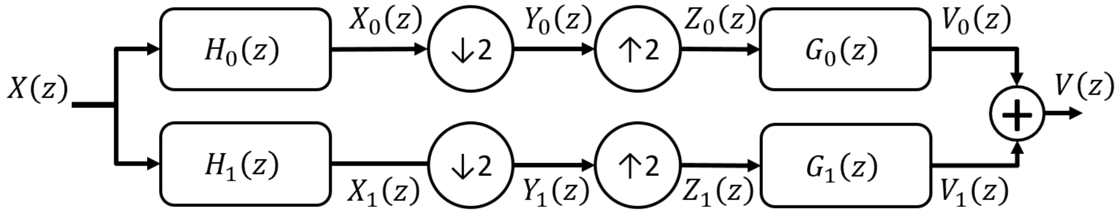
(b) *(8 pts)* Approximate $H_d(s)$ as a discrete-time IIR filter using the impulse invariance method with a sampling rate $T = \pi$. **Determine the resulting z-domain poles and zeros.**

Solution: Poles: $s = j+1, -j+1$

$$\begin{aligned}
H_b(z) &= \frac{5}{1 - e^{(j+1)(\pi)}z^{-1}} + \frac{5}{1 - e^{(-j+1)(\pi)}z^{-1}} \\
&= \frac{5}{1 - e^{-j\pi+\pi}z^{-1}} + \frac{5}{1 - e^{-j\pi+\pi}z^{-1}} \\
&= \frac{5}{1 + e^{\pi}z^{-1}} + \frac{5}{1 + e^{\pi}z^{-1}} \\
&= \frac{10}{1 + e^{\pi}z^{-1}} = \frac{10z}{z + e^{\pi}}
\end{aligned}$$

poles: $z = -e^{\pi}$
zeros: $z = 0$

Question #5: Consider a 2-channel filter bank shown below.



Let the filters be defined by the impulse responses

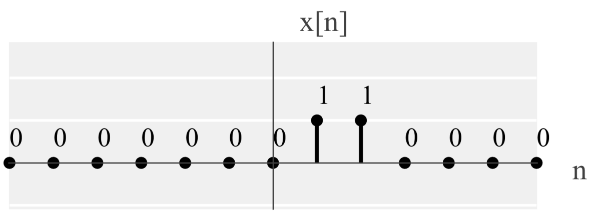
$$h_0[n] = g_0[-n] = \frac{1}{\sqrt{2}}(\delta[n] + \delta[n-1]) \qquad , \qquad h_1[n] = g_1[-n] = \frac{1}{\sqrt{2}}(\delta[n] - \delta[n-1])$$

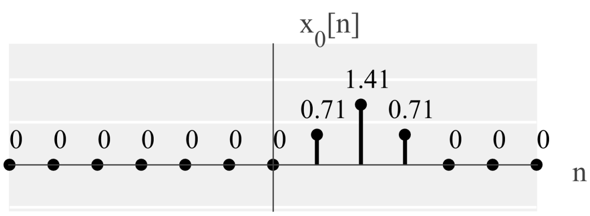
(a) *(8 pts)* Compute $v_0[n]$ (inverse z-transform of $V_0(z)$) for $X(z) = z^{-1} + z^{-2}$.

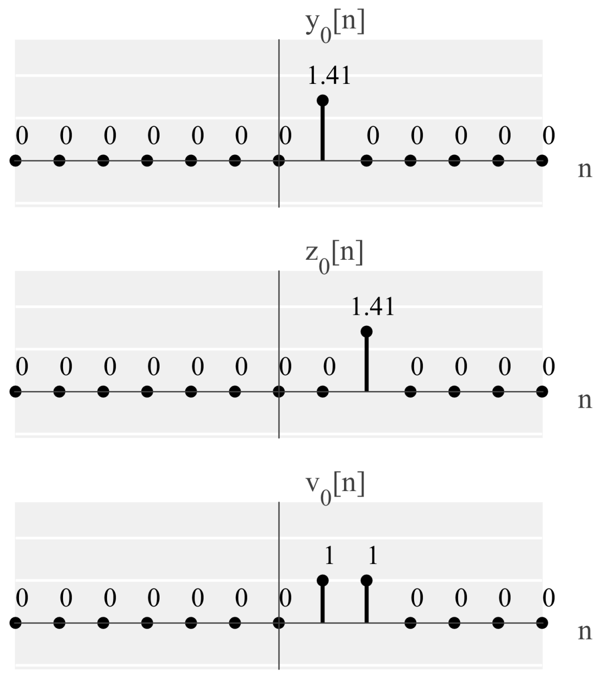
Solution: Solution via z-transform:

$$\begin{aligned}
X(z) &= z^{-1} + z^{-2} \\
H_0(z) &= \frac{1}{\sqrt{2}}(1 + z^{-1}) \\
G_0(z) &= \frac{1}{\sqrt{2}}(z^{+1} + 1) \\
X_0(z) &= \frac{1}{\sqrt{2}}(z^{-1} + z^{-2})(1 + z^{-1}) = \frac{1}{\sqrt{2}}\left(z^{-1} + 2z^{-2} + z^{-3}\right) \\
Y_0(z) &= \frac{1}{2}\left[X_0(z^{1/2}) + X_0(-z^{1/2})\right] \\
&= \frac{1}{2\sqrt{2}}\left[\left(z^{-1/2} + 2z^{-2/2} + z^{-3/2}\right) + \left(-z^{-1/2} + 2z^{-2/2} + -z^{-3/2}\right)\right] = \frac{4}{2\sqrt{2}}z^{-2/2} \\
Z_0(z) &= \frac{2}{\sqrt{2}}z^{-2} \\
V_0(z) &= z^{-1} + z^{-2}
\end{aligned}$$

Solution via time domain:






y0[n]
1.41
0 0 0 0 0 0 0 0 0 0 0 0
n
z0[n]
1.41
0 0 0 0 0 0 0 0 0 0 0 0
n
v0[n]
1 1
0 0 0 0 0 0 0 0 0 0 0
n

(b) *(5 pts)* **(True or False)** Assuming orthogonal filter bank conditions are met, $V(z)$ does **not** change if we switch the downsampling and upsampling operations. **Briefly justify why.**

Solution: False. Short Answer: Switching the downsampling and upsampling operations changes the reconstruction conditions. Switching them cause each operation to cancel each other out. Hence, $V_0(z) = X_0(z)$ and $V_0(z) = X_0(z)$.

Long Answer: Mathematically, our old arrangement above would get

$$\begin{aligned} Y_0(z) &= \frac{1}{2}\left[X_0(z^{1/2}) + X_0(-z^{1/2})\right] \\ Z_0(z) &= \frac{1}{2}\left[X_0(z) + X_0(-z)\right] \\ V_0(z) &= \frac{1}{2}\left[X_0(z) + X_0(-z)\right] G_0(z) \end{aligned}$$

Similarly,

$$V_1(z) = \frac{1}{2}\left[X_1(z) + X_1(-z)\right] G_1(z)$$

With the new arrangement, we will get

$$\begin{aligned} Y_0(z) &= X_0(z^2) \\ Z_0(z) &= \frac{1}{2}\left[X_0((z^{1/2})^2) + X_0((-z^{1/2})^2)\right] \\ &= \frac{1}{2}\left[X_0(z) + X_0(z)\right] = X_0(z) \\ V_0(z) &= X_0(z)G_0(z) \end{aligned}$$

Similarly,

$$V_1(z) = X_1(z)G_1(z)$$

So,

$$\begin{aligned} V(z) &= X_0(z)G_0(z) + X_1(z)G_1(z) \\ &= X_0(z)\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right] \\ &= X_0(z)\left[H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1})\right] \end{aligned}$$

This is a different $V(z)$ than in our original scenario.

(c) *(5 pts)* **(True or False)** Assuming orthogonal filter bank conditions are met, $V(z)$ does **not** change if we time-reverse every filter impulse response. **Briefly justify why.**

Solution: **True. Short Answer:** Switching filters does not change the reconstruction conditions, only the filters. Furthermore, since the right-hand side of the conditions is not dependent on z, changing $z \to z^{-1}$ does not change the result.

Long Answer: The orthogonal filter bank conditions are typically

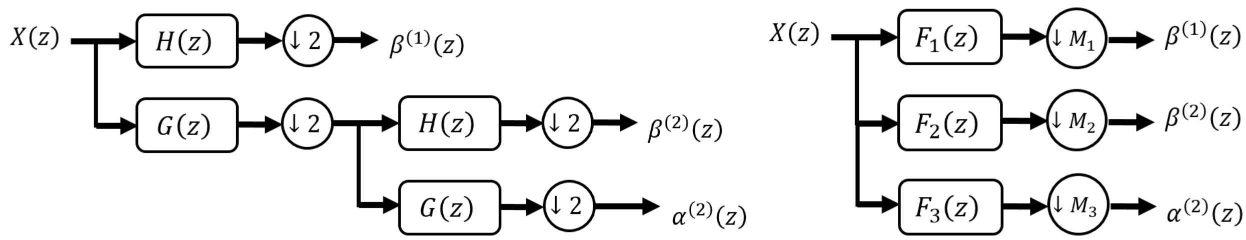
$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$$
$$H_1(z)H_1(z^{-1}) + H_1(-z)H_1(-z^{-1}) = 2$$
$$H_0(z)H_1(z^{-1}) + H_0(-z)H_1(-z^{-1}) = 0$$

The reconstruction condition now turns into

$$H_0(z^{-1})H_0(z) + H_0(-z^{-1})H_0(-z) = 2$$
$$H_1(z^{-1})H_1(z) + H_1(-z^{-1})H_1(-z) = 2$$
$$H_0(z^{-1})H_1(z) + H_0(-z^{-1})H_1(-z) = 0$$

and remains satisfied.

Question #6: Consider the following wavelet bank and filter bank.



Let $H(z)$ and $G(z)$ be defined by the transfer functions:

$$H(z) = 1 \qquad , \qquad G(z) = z^{-1}$$

(a) *(6 pts)* Use the Noble identities to simplify the wavelet bank (left) and represent it as a filter bank (right). Determine M_1, M_2, M_3, $F_1(z)$, $F_2(z)$, $F_3(z)$. Fully simplify.

Solution: $M_1 = 2$, $M_2 = 4$, $M_3 = 4$.

$$F_1(\omega) = H(z) = 1$$
$$F_2(\omega) = G(z)H(z^2) = z^{-1}$$
$$F_3(\omega) = G(z)G(z^2) = z^{-3}$$

(b) *(10 pts)* Let $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 5\delta[n-4]$. Sketch $\beta^{(1)}[n]$, $\beta^{(2)}[n]$, $\alpha^{(2)}[n]$ (the inverse z-transforms of $\beta^{(1)}(z)$, $\beta^{(2)}(z)$, and $\alpha^{(2)}(z)$).

Solution:

$$\beta^{(1)}[n] = \delta[n] + 3\delta[n-1]$$
$$\beta^{(2)}[n] = 2\delta[n-1]$$
$$\alpha^{(2)}[n] = 4\delta[n-1]$$

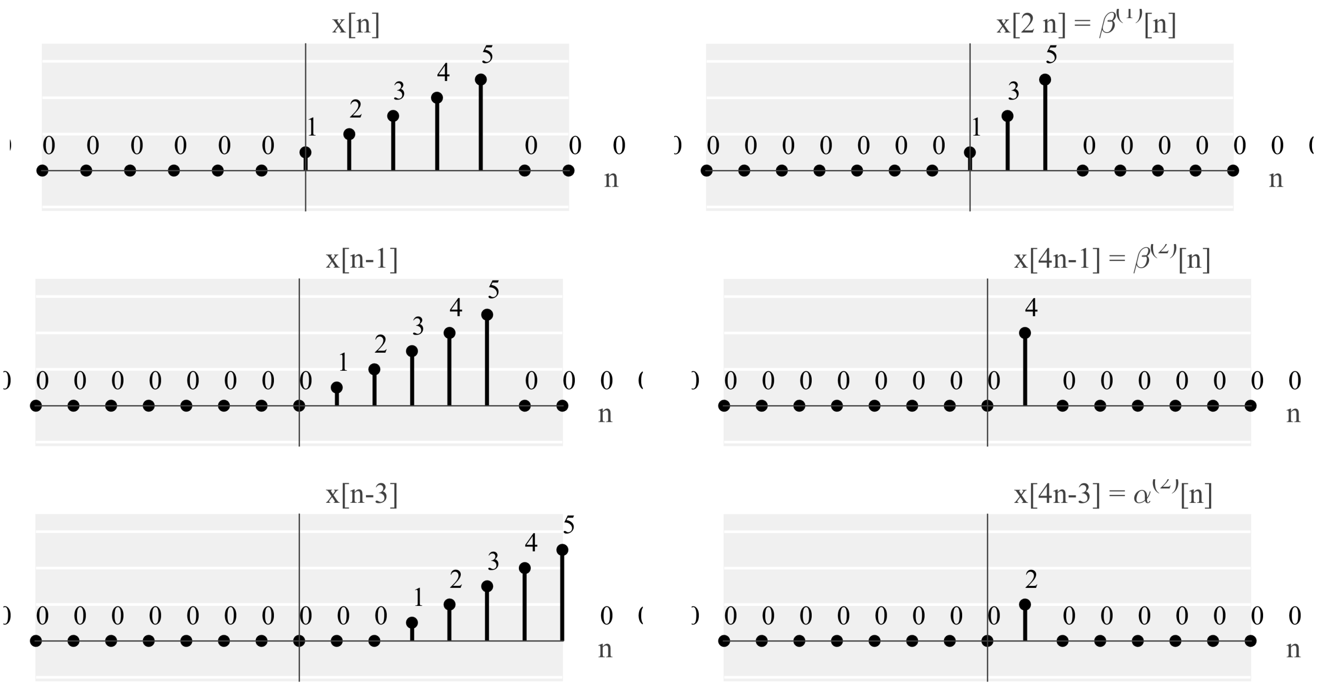


Table of Discrete-Time Fourier Transform Pairs:

$$\text{Discrete-Time Fourier Transform} \quad : \quad X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\text{Inverse Discrete-Time Fourier Transform} \quad : \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega t}\, d\omega \ .$$

$x[n]$	$X(\omega)$	condition
$a^n u[n]$	$\frac{1}{1-ae^{-j\omega}}$	$\lvert a\rvert < 1$
$(n+1)a^n u[n]$	$\frac{1}{(1-ae^{-j\omega})^2}$	$\lvert a\rvert < 1$
$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n]$	$\frac{1}{(1-ae^{-j\omega})^r}$	$\lvert a\rvert < 1$
$\delta[n]$	1	
$\delta[n-n_0]$	$e^{-j\omega n_0}$	
$x[n]=1$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$	
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2\pi k)$	
$e^{j\omega_0 n}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$	
$\cos(\omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k)\}$	
$\sin(\omega_0 n)$	$\frac{\pi}{j} \sum_{k=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi k) - \delta(\omega + \omega_0 - 2\pi k)\}$	
$\sum_{k=-\infty}^{\infty} \delta[n-kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	
$x[n] = \begin{cases} 1 & , \quad \lvert n\rvert \le N \\ 0 & , \quad \lvert n\rvert > N \end{cases}$	$\frac{\sin(\omega(N+1/2))}{\sin(\omega/2)}$	
$\frac{\sin(Wn)}{\pi n} = \frac{W}{\pi}\text{sinc}\left(\frac{Wn}{\pi}\right)$	$X(\omega) = \begin{cases} 1 & , \quad 0 \le \lvert\omega\rvert \le W \\ 0 & , \quad W < \lvert\omega\rvert \le \pi \end{cases}$	

$X(\omega)$ is periodic with period 2π

Table of Discrete-Time Fourier Transform Properties: For each property, assume

$$x[n] \stackrel{DTFT}{\longleftrightarrow} X(\omega) \quad \text{and} \quad y[n] \stackrel{DTFT}{\longleftrightarrow} Y(\omega)$$

Property	Time domain	DTFT domain
Linearity	$Ax[n] + By[n]$	$AX(\omega) + BY(\omega)$
Time Shifting	$x[n - n_0]$	$X(\omega)e^{-j\omega n_0}$
Frequency Shifting	$x[n]e^{j\omega_0 n}$	$X(\omega - \omega_0)$
Conjugation	$x^*[n]$	$X^*(-\omega)$
Time Reversal	$x[-n]$	$X(-\omega)$
Convolution	$x[n] * y[n]$	$X(\omega)Y(\omega)$
Multiplication	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(\theta)Y(\omega - \theta)d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(\omega)$
Accumulation	$\sum_{k=-\infty}^{\infty} x[k]$	$\frac{1}{1-e^{-j\omega}} + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
Frequency Differentiation	$nx[n]$	$j\dfrac{dX(\omega)}{d\omega}$
Parseval's Relation for Aperiodic Signals	$\sum_{k=-\infty}^{\infty} \lvert x[k]\rvert^2$	$\frac{1}{2\pi} \int_{2\pi} \lvert X(\omega)\rvert^2 d\omega$

Table of Z-Transform Pairs:

$$\text{Z-Transform} \quad : \quad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\text{Inverse Z-Transform} \quad : \quad x[n] = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{n-1} \, dz \, .$$

$x[n]$	$X(\omega)$	ROC
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1 - az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| < \|a\|$
$\delta[n]$	1	All z
$\delta[n - n_0]$	z^{-n_0}	All z
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$
$\cos(\omega_0 n) u[n]$	$\frac{1 - z^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$	$\|z\| > 1$
$\sin(\omega_0 n) u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$	$\|z\| > 1$
$a^n \cos(\omega_0 n) u[n]$	$\frac{1 - az^{-1}\cos(\omega_0)}{1 - 2az^{-1}\cos(\omega_0) + a^2 z^{-2}}$	$\|z\| > \|a\|$
$a^n \sin(\omega_0 n) u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1 - a2z^{-1}\cos(\omega_0) + a^2 z^{-2}}$	$\|z\| > \|a\|$

Table of Z-Transform Properties: For each property, assume

$$x[n] \overset{Z}{\longleftrightarrow} X(z) \quad \text{and} \quad y[n] \overset{Z}{\longleftrightarrow} Y(z)$$

Property	Time domain	Z-domain
Linearity	$Ax[n] + By[n]$	$AX(z) + BY(z)$
Time Shifting	$x[n - n_0]$	$X(z)z^{-n_0}$
Z-scaling	$a^n x[n]$	$X(a^{-1}z)$
Conjugation	$x^*[n]$	$X^*(z^*)$
Time Reversal	$x[-n]$	$X(z^{-1})$
Convolution	$x[n] * y[n]$	$X(z)Y(z)$
Differentiation in z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$
Initial Value Theorem	$x[n]$ is causal	$x(0) = \lim_{z\to\infty} X(z)$